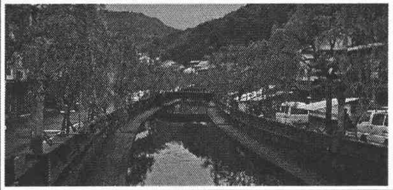
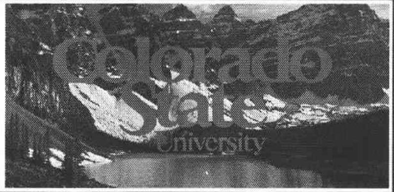
代数幾何学シンポジウム記録
2008年度　pp.143-143



A classification of Lagrangian fibrations by Jacobians

JUSTIN SAWON

Dept of Mathematics, Colorado State University, Fort Collins CO 80523-1874, USA
sawon@math.colostate.edu



ABSTRACT

Let $C \to \mathbb{P}^n$ be a family of reduced and irreducible genus n curves with 'mild singularities', such that the compactified relative Jacobian $X = \overline{\mathrm{Pic}}^1(C/\mathbb{P}^n)$ is a holomorphic symplectic manifold. We prove that X is a Beauville-Mukai integrable system [1] provided the discriminant locus in $\mathbb{P}^n$ has sufficiently large degree. This work appears in preprint **arXiv:0803.1186**.

1 Introduction

An *irreducible holomorphic symplectic manifold* is a compact, simply connected, Kähler manifold X such that $\mathrm{H}^0(X, \Omega^2)$ is generated by a non-degenerate two-form σ. Matsushita [10] proved that the fibres of a non-trivial fibration on X must be Lagrangian with respect to σ, with generic fibre an abelian variety of dimension $n = \frac{1}{2}\dim X$. Hwang [7] proved that the base must be isomorphic to $\mathbb{P}^n$ if X is projective. We call $\pi : X \to \mathbb{P}^n$ a *Lagrangian fibration*.
In [15] the author described how one might use Lagrangian fibrations to classify holomorphic symplectic manifolds up to deformation. It follows from work of Beauville [1], Debarre [2], O'Grady [13], and Rapagnetta [14] that all known examples of holomorphic symplectic manifolds can be deformed to Lagrangian fibrations. The aim of this work is to classify Lagrangian fibrations whose fibres are Jacobians. We say that a family $C \to \mathbb{P}^n$ of curves has *mild singularities* if the total space C is smooth.

Theorem: Let $C \to \mathbb{P}^n$ be a family of reduced and irreducible genus n curves with mild singularities. Suppose that the compactified relative Jacobian $X = \overline{\mathrm{Pic}}^1(C/\mathbb{P}^n)$ is a Lagrangian fibration, and the degree of the discriminant locus $\Delta \subset \mathbb{P}^n$ parametrizing singular curves is greater than $4n + 20$. Then X is a Beauville-Mukai integrable system [1], i.e., the family of curves C is a complete linear system of curves on a K3 surface S, and X can be identified with an irreducible component of the Mukai moduli space of stable sheaves on S. In particular, X is a deformation of the Hilbert scheme $\mathrm{Hilb}^n S$ of n points on S.

In dimension four, a formula of the author [16] can be combined with Guan's bounds [3] on the Chern numbers to verify the lower bound on $\deg\Delta$. We therefore recover a theorem of Markushevich [9] (the $n = 2$ case).
The proof of our theorem closely follows a construction of Hurtubise [5], which uses coisotropic reduction. The main difficulty is in extending Hurtubise's local argument to a global setting. Taking quotients of coisotropic submanifolds is a well-known idea in real symplectic geometry; the author feels it could be further exploited in holomorphic symplectic geometry. For example, Hwang and Oguiso [6] have recently used characteristic foliations to study the structure of singular fibres of Lagrangian fibrations.

2 Coisotropic reduction

Since X is the *degree one* Jacobian the relative Abel-Jacobi map $C \to X$ is well-defined; the image $Y \subset X$ is smooth by the mild singularities hypothesis. Assuming that the restriction $\sigma|_Y$ of the holomorphic symplectic form to Y has rank two everywhere, Hurtubise [5] introduced the following construction. The null directions of $\sigma|_Y$ define a rank $n-1$ foliation F

$$0 \to F \to TY \to TY/F \to 0$$

known as the *characteristic foliation*. The space of leaves $Q = Y/F$ must be a holomorphic symplectic surface, since $\sigma|_Y$ descends to a non-degenerate two-form on Q. The curves in the family C project down to Q.
Hurtubise's argument is local: a family of curves over a small ball in $\mathbb{C}^n$ leads to an open subset Q of an algebraic surface. To obtain a nice space of leaves in a global setting we need compactness (algebraicity) of the leaves. The relevant theorems are due to Miyaoka [12], Bogomolov, and McQuillan (see Kebekus, Solá Conde, and Toma [8]). The key idea is that if Y is covered by curves on which F is ample, then by applying Mori's bend-and-break argument one can produce rational curves which must be contained in the leaves of F.

3 Rational curves in Y

Let ℓ be a generic line in the base $\mathbb{P}^n$ and let Z be the inverse image of ℓ in Y.

$$\begin{array}{ccc} Z & \hookrightarrow & Y \subset X \\ \pi\downarrow & & \downarrow\pi \\ \ell & \hookrightarrow & \mathbb{P}^n \end{array}$$

Then Z is a smooth surface fibred by genus n curves over $\ell \cong \mathbb{P}^1$, with $\deg\Delta$ singular fibres. In particular

$$c_2(Z) = 4 - 4n + \deg\Delta > 24.$$

Matsushita [11] proved that $R^1\pi_*\mathcal{O}_X \cong \Omega^1_{\mathbb{P}^n}$ for a general Lagrangian fibration $\pi : X \to \mathbb{P}^n$. We can use this to show that $R^1\pi_*\mathcal{O}_Y$ is also isomorphic to $\Omega^1_{\mathbb{P}^n}$, and then restrict to ℓ to calculate $R^1\pi_*\mathcal{O}_Z$. Inserting this in the Leray spectral sequence yields

$$h^{0,k}(Z) = \begin{cases} 1 & k = 0, \\ 0 & k = 1, \\ 1 & k = 2. \end{cases}$$

Noether's formula now gives

$$K_Z^2 = 12\chi(\mathcal{O}_Z) - c_2(Z) < 0.$$

If Z were minimal, it would have Kodaira dimension $-\infty$ and hence p_g would vanish. Therefore Z contains at least one (-1)-curve. Varying ℓ in $\mathbb{P}^n$ gives many rational curves in Y.
Consider the following exact sequences over one of these rational curves.

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & T\mathbb{P}^1 & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & F|_{\mathbb{P}^1} & \to & TY|_{\mathbb{P}^1} & \to & TY/F|_{\mathbb{P}^1} & \to & 0 \\ & & & & \downarrow & & & & \\ 0 & \to & N_{\mathbb{P}^1 \subset Z} & \to & N_{\mathbb{P}^1 \subset Y} & \to & N_{Z \subset Y}|_{\mathbb{P}^1} & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

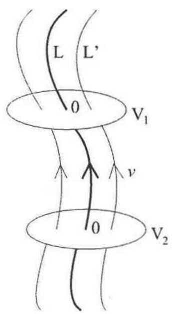
We can show that $F|_{\mathbb{P}^1}$ is isomorphic to $\mathcal{O}(1)^{\oplus(n-2)} \oplus T\mathbb{P}^1$. Thus the rational curve is contained in a leaf of F. Moreover, $F|_{\mathbb{P}^1}$ is ample which implies that all leaves of F are algebraic and rationally connected (see [8]).
Recall that we needed $\sigma|_Y$ to have rank two everywhere so that Y is coisotropic. This is proved along the way: we first define a coherent sheaf F as the kernel of the morphism $TY \to \Omega^1_Y$ given by $\sigma|_Y$. Then we show that F is locally free over a generic rational curve, and the above exact sequences imply that F has rank $n-1$. This proves that $\sigma|_Y$ has rank two at a generic point, and hence everywhere by semi-continuity.

4 The space of leaves

Algebraicity of the leaves implies that they are compact. Holmann [4] proved that a foliation on a Kähler manifold with all leaves compact must be *stable*, which implies that the space of leaves Y/F will be Hausdorff.
Let L be a leaf of the foliation F. A local model for the space of leaves is given by taking a small slice V transverse to the foliation, which meets L at $0 \in V$. The holonomy map is a group homomorphism from $\pi_1(L)$ to the group of automorphisms of V fixing 0, and the holonomy group $H(L)$ is the image of this map. The quotient $V/H(L)$ is a local model for the space of leaves Y/F (see Holmann [4]). In our case both $\pi_1(L)$ and $H(L)$ must be trivial, because rationally connected implies simply connected. Thus the local model is simply V and Y/F is smooth.
It follows that the space of leaves is a smooth compact surface S. Given two local models V_1 and V_2 around the same leaf L, we can find a vector field v along the foliation F whose flow takes V_1 to V_2. The Lie derivative

$$\mathcal{L}_v(\sigma|_Y) = v(d\sigma|_Y) - d(i(v)\sigma|_Y)$$

vanishes, and hence the flow takes $\sigma|_{V_1}$ to $\sigma|_{V_2}$. This defines a (non-degenerate) two-form on $S = Y/F$ which is independent of the choice of local model.



The compact complex surface S therefore admits a holomorphic symplectic structure, so it must be either a K3 or abelian surface.

5 Completion of the proof

A (-1)-curve $\mathbb{P}^1 \subset Z$ will map isomorphically to the line ℓ. Comparing the normal bundle of this $\mathbb{P}^1$ inside a leaf to the normal bundle of ℓ inside $\mathbb{P}^n$ shows that each leaf maps birationally to a hyperplane in $\mathbb{P}^n$. One can show that the curves $C_t \subset Y$ in the family $C \to \mathbb{P}^n$ are everywhere transverse to the leaves of the characteristic foliation, so this birational map is an isomorphism. In particular, a leaf will intersect a curve C_t at most once, and hence each curve C_t maps isomorphically to its image in the space of leaves $S = Y/F$.
This shows that S contains an n-dimensional linear system of genus n curves, so it must be a K3 surface. The family of curves $C \to \mathbb{P}^n$ is therefore a complete linear system of curves on a K3 surface S, and $X = \overline{\mathrm{Pic}}^1(C/\mathbb{P}^n)$ is a Beauville-Mukai integrable system [1].

References

[1] A. Beauville, *Counting rational curves on K3 surfaces*, Duke Math. Jour. **97** (1999), no. 1, 99–108.

[2] O. Debarre, *On the Euler characteristic of generalized Kummer varieties*, Amer. J. Math. **121** (1999), no. 3, 577–586.

[3] D. Guan, *On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four*, Math. Res. Lett. **8** (2001), no. 5-6, 663–669.

[4] H. Holmann, *On the stability of holomorphic foliations*, Analytic functions (Kozubnik, 1979), 192–202, Lecture Notes in Math. **798**, Springer, 1980.

[5] J. Hurtubise, *Integrable systems and algebraic surfaces*, Duke Math. Jour. **83** (1996), no. 1, 19–50.

[6] J.-M. Hwang and K. Oguiso, *Characteristic foliation on the discriminantal hypersurface of a holomorphic Lagrangian fibration*, preprint **arXiv:0710.2376**.

[7] J.-M. Hwang, *Base manifolds for fibrations of projective irreducible symplectic manifolds*, preprint **arXiv:0711.3224**.

[8] S. Kebekus, L. Solá Conde, and M. Toma, *Rationally connected foliations after Bogomolov and McQuillan*, J. Algebraic Geom. **16** (2007), no. 1, 65-81.

[9] D. Markushevich, *Lagrangian families of Jacobians of genus 2 curves*, J. Math. Sci. **82** (1996), no. 1, 3268–3284.

[10] D. Matsushita, *On fibre space structures of a projective irreducible symplectic manifold*, Topology **38** (1999), no. 1, 79–83. Addendum, Topology **40** (2001), no. 2, 431–432.

[11] D. Matsushita, *Higher direct images of dualizing sheaves of Lagrangian fibrations*, Amer. J. Math. **127** (2005), no. 2, 243–259.

[12] Y. Miyaoka, *Deformations of a morphism along a foliation and applications*, Proc. Symp. Pure Math. **46** (1987), 245–268.

[13] K. O'Grady, *Desingularized moduli spaces of sheaves on a K3*, J. Reine Angew. Math. **512** (1999), 49-117.

[14] A. Rapagnetta, *Topological invariants of O'Grady's six dimensional irreducible symplectic variety*, Math. Z. **256** (2007), no. 1, 1–34.

[15] J. Sawon, *Abelian fibred holomorphic symplectic manifolds*, Turkish Jour. Math. **27** (2003), no. 1, 197–230.

[16] J. Sawon, *On the discriminant locus of a Lagrangian fibration*, Math. Ann. **341** (2008), no. 1, 201–221.